ON THE DETERMINATION OF THE ASYMPTOTIC DEVELOPMENTS OF A GIVEN FUNCTION

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1. Introduction. The determination of the asymptotic developments of a given function is usually a problem of considerable difficulty and, when regarded in a general sense, is one for which but fragmentary results exist at the present time. The known determinations appear to be either those for special functions such as Bessel's function $J_n(z)$* or for certain types of integral functions defined either by infinite products or by Maclaurin series.† The importance of such developments is, however, well known. In particular, if $f(z)$ be an integral function of z defined by means of a Weierstrass product,‡ the point $z = \infty$ will in general be essentially singular and there will be no direct means of determining the behavior of $f(z)$ in the neighborhood of this point since the given product is not adapted in form to the study of the function when $|z|$ is large. Such determinations are often important and are supplied as soon as the asymptotic developments are known. Likewise, asymptotic developments in general supply information regarding the behavior of functions in the neighborhood of the point infinity.

The problem, when stated in a precise form and in the one which we shall understand throughout the present paper, may be described as follows: Let $F(z)$ be a given function of the complex variable z defined throughout the finite z plane and such that (a) the point $z = \infty$ is an essentially singular point; and (b) when $|z|$ is sufficiently large and *arg* z lies within some specified sector Λ there exist two functions $f_\lambda(z)$ and $\phi_\lambda(z)$, defined for values of z in Λ and such that for the same values of z

$$(1)\quad F(z)=f_\lambda(z)+\phi_\lambda(z)\left[a_{0,\lambda}+\frac{a_{1,\lambda}}{z}+\frac{a_{2,\lambda}}{z^2}+\cdots+\frac{a_{n-1,\lambda}}{z^{n-1}}+\frac{a_{n,\lambda}+w_{\lambda,n}(z)}{z^n}\right],$$

* Cf. Lommel, *Studien über die Bessel'schen Functionen* § 17 (1868).

† Cf. Barnes, *Philosophical Transactions*, Vol. 199 A, pp. 411–500 (1902); also Vol. 206 A, pp. 249–297 (1906). Each of these memoirs contains an excellent bibliography of the subject. Cf. also Mattson, *Contributions à la Théorie des Fonctions entières.* (Thèse) Upsal, 1905.

‡ Cf. Picard, *Traité d'Analyse*, Vol. II, pp. 140-143.

where $a_{0,\lambda}$, $a_{1,\lambda}$, $\cdots$ $a_{n,\lambda}$ (n arbitrary) are constants and

$$\lim_{|z|=\infty} w_{\lambda,n}(z) = 0.^*$$

To determine the functions $f_\lambda(z)$, $\phi_\lambda(z)$ *and the constants* $a_{0,\lambda}$, $a_{1,\lambda}$, $\cdots$ $a_{n,\lambda}$.†

In the present paper it is proposed to show how the so-called Maclaurin sum-formula may be used in some cases to solve the above problem.‡ For this purpose we shall apply the formula to a variety of special functions $F(z)$. No attempt will be made to obtain theorems of great generality, the belief being that a few illustrations will suffice to enable the reader to apply the method wherever possible for himself. In each of the cases considered only the functions $f_\lambda(z)$, $\phi_\lambda(z)$ and the first one of the constants $a_{0,\lambda}$, $a_{1,\lambda}$, $\cdots$ $a_{n,\lambda}$ which is not equal to zero are determined since these three determinations constitute what is essential to the study of the behavior of the function for large values of $|z|$. The method, however, permits equally of the determination of $a_{n,\lambda}$ when $n = 0, 1, 2, 3, \cdots$.

2. The Maclaurin Sum-Formula. The form in which we shall take the sum-formula above referred to is as follows:§ Let $f(x)$ be a function of the real variable x which together with its first $2k$ derivatives ($k \geqq 1$) is finite and continuous throughout the interval $x \geqq 0$.‖ Also, let

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* It may be remarked that a function $F(z)$ may frequently be written as a linear combination of expressions of the form (1) when $|z|$ is sufficiently large and *arg* z is properly chosen, as in the case of Bessel's function $J\ (z)$ (cf. Lommel l. c.). In such cases the resulting development is likewise said to be asymptotic, but in the present paper we shall confine ourselves to the primary form (1).

† Frequently the proof of the *existence* of (b) constitutes a separate preliminary problem. This is the case for the functions considered in the present paper where both the existence and determination of the developments (1) are considered.

‡ This formula likewise constitutes the central feature of the method employed by Barnes in the first of the memoirs cited above (l. c. p. 444). The form in which it there occurs is essentially different, however, from that which we adopt, the latter appearing preferable because it takes into account the *remainder*, thereby rendering possible a greater degree of rigor and clearness in the deductions; also because certain restrictions present in the former case may be dispensed with, reference being here made especially to those imposed upon $\phi(z)$ (l. c., §41).

§ Cf. Markoff, *Differenzenrechnung* (Leipzig, 1896), pp. 98, 99, 132. Throughout we shall take $h = 1$.

‖ No conditions are explicitly stated by Markoff as regards the function $f(x)$, but it appears from the analysis of pp. 112, 113, 114 that the conditions which we here impose are sufficient.

$$\phi_q(t) = \frac{t^q}{q!} + A_1 \frac{t^{q-1}}{(q-1)!} + A_2 \frac{t^{q-2}}{(q-2)!} + \cdots + A_{q-1}t,$$

where $$A_1 = -\tfrac{1}{2},\ A_3 = A_5 = A_7 = \cdots = A_{2q+1} = 0$$

and $$A_{2q} = \frac{(-1)^{q-1}B_q}{(2q)!},$$

B_q representing the qth Bernoulli number.

Then, whenever the series

$$(2) \qquad \Omega_k(m) = \sum_{n=1}^{n=\infty} \int_0^1 f^{(2k)}(m+n-t)\phi_{2k}(t)\,dt \qquad (m \geqq a)$$

converges we may write

$$(3) \quad \sum_{x=a}^{x=m-1} f(x) = C_k + \int_a^m f(x)\,dx + A_1 f(m) + A_2 f'(m) + A_4 f'''(m) + \cdots + A_{2k-2} f^{(2k-3)}(m) + \Omega_k(m), \qquad (a \geqq 0)$$

where C_k is a constant (independent of m) defined by the equation

$$C_k = -[A_1 f(a) + A_2 f'(a) + A_4 f'''(a) + \cdots + A_{2k-2} f^{(2k-3)}(a) + \Omega_k(a)].$$

For the important case in which $k = 1$ the formula becomes

$$(4) \qquad \sum_{x=a}^{x=m-1} f(x) = C + \int_a^m f(x)\,dx - \tfrac{1}{2} f(m) + \Omega_1(m),$$

where $$C = \tfrac{1}{2} f(a) - \Omega_1(a).$$

3. Application to the Study of Asymptotic Developments. *Example* 1. *To obtain asymptotic developments for the function*

$$F(z) = \sum_{n=0}^{n=\infty} \frac{1}{(2n+1)^2 + z^2}.$$

We here choose a function which, by virtue of the well known formula

$$\frac{\tan z}{2z} = \sum_{n=0}^{n=\infty} \frac{1}{\frac{(2n+1)^2}{4}\pi^2 - z^2}$$

may readily be evaluated in the form

$$F(z) = \frac{\pi}{4z} \frac{e^{\frac{\pi z}{2}} - e^{-\frac{\pi z}{2}}}{e^{\frac{\pi z}{2}} + e^{-\frac{\pi z}{2}}},$$

thus enabling us to compare the results obtained with known facts.

Considering at first that z is real but different from zero, let us place

$$f(x) = \frac{1}{(2x+1)^2 + z^2}, \qquad k = 1,\ a = 0.$$

Then the series (2) is convergent. In fact, upon observing that $\phi_2(t)$ is negative for all values of t between $t = 0$ and $t = 1$, we may apply the first law of the mean for integrals and write the nth term of the series—viz.:

$$\int_0^1 f^{(2)}(m+n-t)\phi_2(t)\,dt,$$

where the indicated second derivative is with respect to m – in the form

$$f^{(2)}(m+n-\theta)\int_0^1 \phi_2(t)\,dt = -\tfrac{1}{12} f^{(2)}(m+n-\theta) \qquad (0 < \theta < 1)$$

from which the indicated result readily appears.

Likewise it appears that the nth term of the series (2) approaches the limit zero when $m = \infty$ and hence that $\lim\limits_{m=\infty} \Omega_1(m) = 0$.

Upon employing formula (4) and placing $m = \infty$; also noting that $\lim\limits_{m=\infty} f(m) = 0$, we thus obtain

$$F(z) = \int_0^1 \frac{dx}{(2x+1)^2 + z^2} + \frac{1}{2(1+z^2)} + R(z)$$

where

$$(5) \qquad R(z) = -\Omega_1(0) = -\sum_{n=1}^{n=\infty} \int_0^1 \left\{ \frac{d^2}{dx^2}\left[\frac{1}{(2x+1)^2 + z^2}\right]\right\}_{x=n-t} \phi_2(t)\,dt.$$

We note also that since each term of the series (5) when multiplied by z vanishes when $z = \pm\infty$, we shall have $\lim\limits_{z=\pm\infty} zR(z) = 0$.

Moreover, if in particular z is *positive* we may write

$$\int_0^1 \frac{dx}{(2x+1)^2+z^2} = \frac{1}{2z}\left[\text{arc tan}\,\frac{2x+1}{z}\right]_{x=0}^{x=\infty} = \frac{\pi}{4z} - \frac{1}{2z}\,\text{arc tan}\,\frac{1}{z},$$

so that for large positive values of z we obtain

$$(6) \qquad F(z) = \frac{\pi}{4z} + \frac{\epsilon(z)}{z}\,; \qquad \lim_{z=+\infty} \epsilon(z) = 0.$$

This last result may now be generalized to all values of z, real or complex, lying in any sector Λ whose vertex is at the point $z = 0$ and whose bounding lines lie within the right half of the z plane, the neighborhood of the point $z = 0$ being always understood to be excluded. To see this we first note that the series appearing in (5) not only converges, as we have indicated, for values of z which are real and different from zero, but for all values of z in Λ the convergence is likewise seen to exist and to be *uniform*. Moreover, each term of $R(z)$ is analytic throughout Λ. Thus it follows that $R(z)$ is itself analytic in this region.* Whence, by taking Λ so large that it includes the portion of the positive real axis in which (6) holds, the two members of the same equation come to represent two functions of the complex variable z, each analytic throughout Λ and coinciding when z is real, and therefore coinciding throughout Λ. Furthermore, since for values of z in Λ each term of $R(z)$ when multiplied by z approaches the limit zero when $|z| = \infty$, it follows that $\lim_{|z|=\infty} zR(z) = 0$ from which we conclude that for the same values of z we may write $\lim_{|z|=\infty} \epsilon(z) = 0$.

On the other hand, when z is real and negative we obtain in like manner

$$F(z) = -\frac{\pi}{4z} + \frac{\epsilon(z)}{z}\,; \qquad \lim_{z=-\infty} \epsilon(z) = \lim_{z=+\infty} \epsilon(z) = 0$$

and by reasoning as before we find that this equation holds true for all values of z ($z = 0$ excluded) within any sector lying in the *left* half of the z plane, it being understood that $\lim_{z=-\infty}$ is then replaced by $\lim_{|z|=\infty}$.

Thus, in summary we reach the following conclusion:

According as $-\dfrac{\pi}{2} + \eta \gtreqless \arg z \gtreqless \dfrac{\pi}{2} - \eta$ *or* $\dfrac{\pi}{2} + \eta \gtreqless \arg z \gtreqless \dfrac{3\pi}{2} - \eta$, η

† Cf. Osgood, *Encyklopädie*, II 2, p. 21.

being arbitrarily small and positive, we may write when $|z| > 0$

$$F(z) = \begin{cases} \dfrac{\pi}{4z}\Big[1 + \epsilon(z)\Big] \\ -\dfrac{\pi}{4z}\Big[1 - \epsilon(z)\Big] \end{cases}; \qquad \lim_{|z|=\infty} \epsilon(z) = 0.$$

For the two sectors above mentioned the function $F(z)$ thus possesses asymptotic developments for which, in the language of §1, we may take respectively $f_1(z) = 0$, $\phi_1(z) = 1$, $a_{0,1} = 0$, $a_{1,1} = \frac{\pi}{4}$ and $f_2(z) = 0$, $\phi_2(z) = 1$, $a_{0,2} = 0$, $a_{1,2} = -\frac{\pi}{4}$. In case more of the coefficients $a_{n,1}$ or $a_{n,2}$ are desired they may be obtained by using larger values of k in applying (3).

4. *Example* 2. The preceding result may readily be generalized as follows: Let

$$F(z) = \sum_{n=0}^{n=\infty} \frac{\mu_n}{\lambda_n + z^2} \tag{7}$$

where λ_n, μ_n when considered throughout the continuous domain $n \geqq 0$ are such that (a) *both are continuous*; (b) *both possess continuous first and second derivatives which remain less in absolute value than a constant*; (c) $\lim_{n=\infty} \lambda_n = +\infty$, $\left|\frac{\lambda_n}{n^p}\right|$ *< a constant, p being a constant* $> \frac{1}{2}$; (d) $|\mu_n|$ *< a constant.*

The immediate application of (4) then shows that when z is real but different from zero we may write

$$F(z) = \int_0^\infty \frac{\mu_x dx}{\lambda_x^2 + z^2} + \frac{1}{2}\,\frac{\mu_0}{(\lambda_0^2 + z^2)} + R(z)$$

where $\lim_{z=\pm\infty} zR(z) = 0$. If we now add the fifth hypothesis that (e) *for all values of n sufficiently large* ($n \geqq n_0 =$ *a constant*) *the first derivative* λ_n' *of* λ_n *shall be positive and always greater than a constant different*

from zero, it appears directly that the integral

$$\int_0^\infty \frac{\mu_x dx}{\lambda_x^2 + z^2}$$

will vanish like $\frac{1}{z}$ when $z = \pm \infty$. In fact we shall then have

$$\int_0^\infty \frac{\mu_x dx}{\lambda_x^2 + z^2} = \int_0^{n_0} \frac{\mu_x dx}{\lambda_x^2 + z^2} + \int_{n_0}^\infty \frac{\mu_x dx}{\lambda_x^2 + z^2}$$

of which the first term in the second member vanishes like $\frac{1}{z^2}$ when $z = \pm \infty$, while the second may be written in the form

$$\theta L \int_{n_0}^\infty \frac{\lambda_x' dx}{\lambda_x^2 + z^2} = \frac{\theta L}{z}\left[\text{arc tan } \frac{\lambda_x}{z}\right]_{x=n_0}^{x=\infty}$$

where L is the greatest value taken by $\frac{|\mu_n|}{\lambda_n'}$ when $n \geqq n_0$ and where $-1 < \theta < 1$.

In particular, when z is positive we may therefore write

$$F(z) = \frac{a}{z}\Big[1 + \epsilon(z)\Big]; \qquad \lim_{z=+\infty} \epsilon(z) = 0$$

where

$$a = \lim_{z=+\infty} z \int_0^\infty \frac{\mu_x dx}{\lambda_x^2 + z^2}.$$

We may now carry out the generalizations of this result as in the preceding example, thus reaching the following conclusion:

According as $-\frac{\pi}{2} + \eta \leqq \arg z \leqq \frac{\pi}{2} - \eta$ *or* $\frac{\pi}{2} + \eta \leqq \arg z \leqq \frac{3\pi}{2} - \eta$, η *being arbitrarily small and positive, the function* $F(z)$ *defined by* (7) *may be put into the form*

$$F(z) = \begin{cases} \dfrac{a}{2}\Big[1 + \epsilon(z)\Big] \\[2ex] -\dfrac{a}{2}\Big[1 - \epsilon(z)\Big] \end{cases}; \qquad a = \lim_{z=+\infty} z \int_0^\infty \frac{\mu_x dx}{\lambda_x^2 + z^2}; \qquad \lim_{|z|=\infty} \epsilon(z) = 0$$

it being assumed throughout that $|z| > 0$ *and that the above indicated conditions* (a), (b), (c), (d) *and* (e) *are satisfied.*

This result is evidently of especial value in all cases where the integral

$$\int_0^\infty \frac{\mu_x dx}{\lambda_x^2 + z^2}$$

is either known or is capable of easy calculation, as in example 1.

It may be added that in order to be assured by the method that the function $\epsilon(z)$ is developable in the precise form called for by (1), where n is supposed to be *arbitrary*, it would be necessary to extend condition (b) to derivatives of *all* orders.

5. *Example* 3. *To obtain asymptotic developments for the function*

$$F(z) = \prod_{n=1}^{n=\infty} \left[1 + \frac{z^2}{n^2}\right].*$$

As in example 1, this function may be evaluated beforehand and is equal to

$$\frac{e^{\pi z} - e^{-\pi z}}{2\pi z},$$

thus furnishing a check on our subsequent results.

We begin by writing

$$(8) \qquad \log F(z) = \sum_{n=1}^{n=\infty} \log\left[1 + \frac{z^2}{n^2}\right]$$

$$= \lim_{m=\infty} \left[\sum_{n=1}^{n=m-1} \log(n^2 + z^2) - 2 \sum_{n=1}^{n=m-1} \log n\right].$$

From the familiar asymptotic expansion for $\log\{(m-1!)\}$† we have at once

$$(9) \qquad -2 \sum_{n=1}^{n=m-1} \log n = -2 \log\{(m-1)!\}$$

$$= -\log 2\pi - 2(m - \tfrac{1}{2})\log m + 2m + \omega_1(m)\,; \lim_{m=\infty} \omega_1(m) = 0\,.$$

We proceed to apply formula (4) to the first summation in the last member of (8), taking for this purpose $f(x) = \log(x^2 + z^2)$ and supposing for the

* Cf. Barnes l. c. (first memoir) § 50.

† Cf. Bromwich, *Infinite Series* (1908), §179.

present that z is real but different from zero. The formula may be applied since the series (2) becomes

$$\Omega_1(m) = \sum_{n=1}^{n=\infty} \int_0^1 \left\{ \frac{d^2}{dx^2} \log(x^2 + z^2) \right\}_{x=m+n-t} \phi_2(t)\,dt$$

and this is readily seen to be convergent when we apply the first law of the mean for integrals to its nth term (cf. example 1). Likewise we see that $\lim_{m=\infty} \Omega_1(m) = 0$.

Formula (4) thus gives

$$(10) \qquad \sum_{n=1}^{n=m-1} \log(n^2 + z^2) = \int_1^m \log(m^2 + z^2)\,dm - \tfrac{1}{2}\log(m^2 + z^2) + \Omega_1(m) + \\ + \tfrac{1}{2}\log(1 + z^2) + R(z)$$

where

$$(11) \qquad R(z) = -\Omega_1(1) = -\sum_{n=1}^{n=\infty} \int_0^1 \left\{ \frac{d^2}{dx^2} \log(x^2 + z^2) \right\}_{x=n-t+1} \phi_2(t)\,dt.$$

By combining relations (8), (9) and (10) and then making use of the relation

$$\int_1^m \log(m^2 + z^2)\,dm = m\log(m^2 + z^2) - 2m + 2z \text{ arc tan} \frac{m}{z} \\ - \log(1 + z^2) + 2 - 2z \text{ arc tan} \frac{1}{z},$$

we obtain

$$\log F(z) = -\log 2\pi - \tfrac{1}{2}\log(1 + z^2) - 2z \text{ arc tan } \frac{1}{z} + 2 + R(z) + \\ \lim_{m=\infty} \left[(m - \tfrac{1}{2})\log\left(1 + \frac{z^2}{m^2}\right) + 2z \text{ arc tan } \frac{m}{z} + \omega_1(m) + \Omega_1(m) \right].$$

Moreover, we have

$$\lim_{m=\infty} (m - \tfrac{1}{2}) \log\left(1 + \frac{z^2}{m^2}\right) = 0, \qquad \lim_{m=\infty} \omega_1(m) = 0, \qquad \lim_{m=\infty} \Omega_1(m) = 0$$

and, supposing at first that z is *positive*, we shall have

$$\lim_{m=\infty} 2z \text{ arc tan } \frac{m}{z} = \pi z$$

and hence

$$(12) \quad \log F(z) = -\log 2\pi z + \pi z - \tfrac{1}{2}\log\left(1+\frac{1}{z^2}\right) + 2\left(1 - z \operatorname{arc}\tan\frac{1}{z}\right) + R(z).$$

This result may now be generalized, as in example 1, for all values of z ($z = 0$ excluded) for which $-\frac{\pi}{2} + \eta \gtreqless arg\, z \gtreqless \frac{\pi}{2} + \eta$, and for such values it appears directly from (11) that $\lim_{|z|=\infty} R(z) = 0$. Also, for the same values of z we have

$$\lim_{|z|=\infty} \log\left(1+\frac{1}{z^2}\right) = 0, \qquad \lim_{|z|=\infty}\left(1 - z \operatorname{arc}\tan\frac{1}{z}\right) = 0,$$

and hence

$$\log F(z) = -\log 2\pi z + \pi z + \eta(z); \quad \lim_{|z|=\infty} \eta(z) = 0.$$

On the other hand, when z is negative we have, instead of (12),

$$\log F(z) = -\log(-2\pi z) - \pi z - \tfrac{1}{2}\log\left(1+\frac{1}{z^2}\right) + 2\left(1 - z \operatorname{arc}\tan\frac{1}{z}\right) + R(z).$$

Thus we reach the following result:

According as $-\frac{\pi}{2} + \eta \gtreqless arg\, z \gtreqless \frac{\pi}{2} - \eta$ *or* $\frac{\pi}{2} + \eta \gtreqless arg\, z \gtreqless \frac{3\pi}{2} - \eta$, *$\eta$ being arbitrarily small and positive, we may write when* $|z| > 0$

$$F(z) = \begin{cases} \dfrac{e^{\pi z}}{2\pi z}\Big[1 + \epsilon(z)\Big] \\ -\dfrac{e^{-\pi z}}{2\pi z}\Big[1 - \epsilon(z)\Big] \end{cases} \qquad \lim_{|z|=\infty} \epsilon(z) = 0.$$

6. *Example* 4. In generalization of the last result we shall merely point out certain consequences of the method as applied to the function

$$F(z) = \prod_{n=1}^{n=\infty}\left[1 + \frac{z^2}{\lambda_n^2}\right]$$

where λ_n when considered throughout the continuous domain $n \geqq 1$ satisfies the following conditions: (a) *never vanishes*; (b) *is continuous*; (c) *has*

continuous first and second derivatives which remain less in absolute value than a constant; *and* (*d*) $\lim_{n=\infty} \frac{\lambda_n}{n} =$ *a constant* $\neq 0$.

We begin by writing

$$(13) \qquad \log F(z) = \lim_{m=\infty} \left[\sum_{n=1}^{n=m-1} \log (\lambda_n^2 + z^2) - 2 \sum_{n=1}^{n=m-1} \log \lambda_n \right].$$

From (4) we have

$$(14) \quad -2 \sum_{n=1}^{n=m-1} \log \lambda_n = c - \int_1^m \log \lambda_m^2 dm + \log \lambda_m + \omega_1(m)\,; \quad \lim_{m=\infty} \omega_1(m) = 0$$

where

$$(15) \qquad c = -\log \lambda_1 + 2 \sum_{n=1}^{n=\infty} \int_0^1 \left\{ \frac{d^2}{dx^2} \log \lambda_x \right\}_{x=n-t+1} \phi_2(t) dt.$$

Again, if z be real but different from zero, the application of (4) to the first summation on the right in (13) gives

$$(16) \qquad \sum_{n=1}^{n=m-1} \log (\lambda_n^2 + z^2) = \int_1^m \log (\lambda_m^2 + z^2) dm - \tfrac{1}{2} \log (\lambda_m^2 + z^2)$$

$$+ \Omega_1(m) + \tfrac{1}{2} \log (\lambda_1^2 + z^2) + R(z)\,; \qquad \lim_{m=\infty} \Omega_1(m) = 0$$

where

$$(17) \qquad R(z) = -\sum_{n=1}^{n=\infty} \int_0^1 \left\{ \frac{d^2}{dx^2} \log (\lambda_x^2 + z^2) \right\}_{x=n-t+1} \phi_2(t)\, dt.$$

Moreover, by an integration by parts we obtain

$$(18) \quad \int_1^m \log (\lambda_m^2 + z^2) dm = m \log (\lambda_m^2 + z^2) - \log (\lambda_1^2 + z^2)$$

$$- 2 \int_1^m \frac{m \lambda_m \lambda_m' dm}{\lambda_m^2 + z^2}$$

where λ_m' denotes the derivative of λ_m with respect to m.

Let us now make use of (14) and (15) in relation (13) and subsequently expand by means of (18) the second term in the second member of (14) and the first term in the second member of (16).

Upon placing $m = \infty$ in the result we obtain

$$\log F(z) = K - \tfrac{1}{2}\log(\lambda_1^2 + z^2) + 2z^2\int_1^\infty \frac{m\lambda_m' dm}{\lambda_m(\lambda_m^2 + z^2)} + R(z)$$

where $K = c + 2\log\lambda_1$.

If we now introduce the additional hypothesis that *for all values of* n *greater than some constant* n_0 *we shall have* $\lambda_n' \geqq 0$, we find directly in the manner already indicated in example 2 that according as z is positive or negative we shall have

$$\log F(z) = \begin{cases} K - \log z + 2a_2 z + \epsilon(z) \\ K - \log(-z) - 2a_2 z - \epsilon(z) \end{cases}; \quad \lim_{z=+\infty}\epsilon(z) = \lim_{z=-\infty}\epsilon(z) = 0$$

where

$$a_2 = 2\lim_{z=+\infty} z\int_1^\infty \frac{m\lambda_m' dm}{\lambda_m(\lambda_m^2 + z^2)} \tag{19}$$

and, if we now proceed as in the previous examples, we may generalize this result into the following:

According as $-\frac{\pi}{2} + \eta \leqq \arg z \leqq \frac{\pi}{2} - \eta$ *or* $\frac{\pi}{2} + \eta \leqq \arg z \leqq \frac{3\pi}{2} - \eta$,

η *being arbitrarily small and positive, we may write when* $|z| > 0$

$$F(z) = \begin{cases} \dfrac{a_1 e^{a_2 z}}{z}\Big[1 + \epsilon(z)\Big] \\ \dfrac{-a_1 e^{-a_2 z}}{z}\Big[1 - \epsilon(z)\Big] \end{cases}; \qquad \lim_{|z|=\infty}\epsilon(z) = 0$$

where $a_1 = e^K$ *and* a_2 *are constants.*

As a noteworthy special consequence of this result, obtained from it by making the substitution $z = iz$ ($i = \sqrt{-1}$) and expanding the resulting exponential functions by De Moivre's theorem, we note the following:

Let $\Phi(z)$ *be any function of the complex variable* z *expressible in the form*

$$\Phi(z) = \prod_{n=1}^{n=\infty}\left[1 - \frac{z^2}{\lambda_n^2}\right]$$

in which λ_n *when considered throughout the continuous domain* $n \geqq 1$ *satisfies the following conditions:* (*a*) *never vanishes;* (*b*) *is continuous;* (*c*) *has*

continuous first and second derivatives of which the first eventually becomes and remains positive, while both always remain less in absolute value than a constant; (*d*) $\lim_{n=\infty} \frac{\lambda_n}{n}$ = *a constant different from zero.*

According as $\eta \gtreqless \arg z \gtreqless \pi - \eta$ *or* $-\pi + \eta \gtreqless \arg z \gtreqless -\eta$, η *being arbitrarily small and positive, we may write when* $|z| > 0$

$$\Phi(z) = \begin{cases} \frac{a_1}{z}[\sin a_2 z - i \cos a_2 z][1 + \epsilon(z)] \\ \frac{a_1}{z}[\sin a_2 z + i \cos a_2 z][1 - \epsilon(z)] \end{cases}; \qquad \lim_{|z|=\infty} \epsilon(z) = 0$$

where a_1 *and* a_2 *are constants.*

7. It will be readily perceived that the method employed in the preceding examples has a wide field of applicability. While we shall not attempt to carry the subject further, it may be noted that information may, in general, be thus obtained concerning the behavior for large values of $|z|$ of any function $F(z)$ defined by a series of the form

$$F(z) = \sum_{n=a}^{n=\infty} f_n(z)$$

when, by taking k sufficiently large, it can be shown that the series

$$\Omega_k(m) = \sum_{n=1}^{n=\infty} \int_0^1 \left\{ \frac{d^{2k}}{dx^{2k}} f_x(z) \right\}_{x=m+n-t} \phi_{2k}(t)\,dt; \qquad m \geqq a$$

is convergent for all values of z in a certain real region, while for the same values of z it can be shown also that $\lim_{m=\infty} \Omega_k(m) = 0$. In such cases we may apply at once formula (3) after which, upon letting $m = \infty$, we obtain an expression for $F(z)$ which yields the desired asymptotic behavior when z is confined to the preassigned real region. By studying the properties of the same expression, we may usually generalize the result so as to obtain the asymptotic behavior for one or more sectors of the z complex plane.

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